An ELLIPSE could be accurately described as a circle that has been stretched or compressed by a constant ratio towards a diameter of a circle. A circle is actually a special type of ellipse. More explicitly, an ELLIPSE is the locus of points whose distance from two focal points is constant. To better understand what this means consider the following:

Paste or tape a coordinate grid to a piece of cardboard. Put two push pins in the grid at (−4, 0) and (4, 0). These pins will represent the focal points. Next tie a string to each push pin such that the length of the string between the two pins is approximately 10 units long (using the unit length of the coordinate grid). Finally, take a pencil and stretch out the string until the string is straightened and trace out all of the places the string allows the pencil to move with the string remaining taut. This will ensure that the combined distance from each focal point will always be a total of 10 units long.

Rather than always having a fixed diameter an oblong ellipse can be described as having a major axis (the longest diameter) and a minor axis (the shortest diameter). The endpoints of the major axis are considered the vertices of an ellipse and the endpoints of the minor axis are consider the co-vertices.

1. Determine the length of the major & minor axis. List the coordinates of vertices and co-vertices of the following ellipses.

A. Major Axis Length: [Diagram]

Minor Axis Length: [Diagram]

Vertices: [Diagram]

Co-Vertices: [Diagram]

B. Major Axis Length: [Diagram]

Minor Axis Length: [Diagram]

Vertices: [Diagram]

Co-Vertices: [Diagram]
Rather than always having a fixed radius an oblong ellipse can be described as having a major radius/semi-major axis (the longest radius) and a minor radius/semi-minor axis (the shortest radius). The major radius is commonly denoted by the variable ‘a’ and the minor radius is commonly denoted by the variable ‘b’. These different radii are used to write the equation of the ellipse in standard form.

**Standard Form of an ELLIPSE centered at the origin.**

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{OR} \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1
\]

**‘a’ must always be the largest radius**

The Equation of the ELLIPSE shown at the left.

\[
\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1 \quad \text{OR} \quad \frac{x^2}{25} + \frac{y^2}{9} = 1
\]

**Standard Form of an ELLIPSE**

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \text{OR} \quad \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1
\]

**‘a’ must always be the largest radius**

The Equation of the ELLIPSE shown at the left.

\[
\frac{(x-(-3))^2}{2^2} + \frac{(y-1)^2}{3^2} = 1 \quad \text{OR} \quad \frac{(x+3)^2}{4} + \frac{(y-1)^2}{9} = 1
\]

2. Determine the equations of the following ellipses in standard form (list the vertices and co-vertices).

A. \[
\frac{x^2}{4} + \frac{y^2}{16} = 1
\]

B. \[
\frac{(x+2)^2}{9} + \frac{y^2}{1} = 1
\]

C. \[
\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1
\]
3. Graph the following equations and list the center and radius of each ellipse.

A. \( \frac{x^2}{25} + \frac{y^2}{4} = 1 \)

- Center: \((0, 0)\)
- Vertices: \((\pm 5, 0)\)
- Co-vertices: \((0, \pm 2)\)

B. \( \frac{(x-2)^2}{9} + \frac{(y+3)^2}{1} = 1 \)

- Center: \((2, -3)\)
- Vertices: \((5, -3)\) and \((-1, -3)\)
- Co-vertices: \((2, -6)\) and \((2, 0)\)

C. \( \frac{(x-1)^2}{36} + \frac{(y-2)^2}{9} = 1 \)

- Center: \((1, 2)\)
- Vertices: \((7, 2)\) and \((-5, 2)\)
- Co-vertices: \((1, 5)\) and \((1, -1)\)

4. Complete the square to put each of the following ellipses in standard form. Then, list the center, vertices, and co-vertices. Finally, graph each ellipse.

\(16x^2 + 4y^2 - 32x = 48\)

- Complete the square:
  \[ 16(x^2 - 2x + 1) + 4y^2 = 64 \]
  \[ 16(x - 1)^2 + 4y^2 = 64 \]
  \[ \frac{(x-1)^2}{4} + \frac{y^2}{16} = 1 \]
- Center: \((1, 0)\)
- Vertices: \((5, 0)\) and \((-3, 0)\)
- Co-vertices: \((1, 4)\) and \((1, -4)\)

\(9x^2 + y^2 + 54x - 4y = -76\)

- Complete the square:
  \[ 9(x^2 + 6x + 9) + y^2 - 4y + 4 = -76 \]
  \[ 9(x+3)^2 + (y-2)^2 = \frac{9}{7} \]
- Center: \((-3, 2)\)
- Vertices: \((-2, 2)\) and \((-4, 2)\)
- Co-vertices: \((-3, 1)\) and \((-3, 3)\)
Finding the focal points algebraically, requires the use of the Pythagorean Theorem. First, consider the constant distance found between any point on the ellipse and the two focal points is equal to the length of the major axis.

In the above example, you can see that \( m + n = 10 \) which is also the length of the major axis. Now, if we position the ‘string’ at one of the co-vertices then a right triangle is formed which is shown below. The hypotenuse is half the length of the major axis which would equal to the length of the major radius. One of the legs of the right triangle is the minor radius and the other leg is represented by the distance from the center of the ellipse to a focal point. It may seem a little out of order but you can see based on the defined variables we could state the following.

The focal points are always located on the major axis and are a distance of ‘c’ units away from the center of the ellipse where ‘c’ is defined by the equation:

\[
\begin{align*}
  b^2 + c^2 &= a^2 \quad &\text{OR} & \quad a^2 - b^2 &= c^2 \\
  \therefore \quad a' \text{ is the length of the major radius and 'b' the minor radius} 
\end{align*}
\]

These equations correspond to the ellipse shown at the left.

\[
\begin{align*}
  3^2 + 4^2 &= 5^2 \quad &\text{OR} & \quad 5^2 - 3^2 &= 4^2 
\end{align*}
\]

6. Find the focal points of each ellipse shown below.

A. \[
\begin{align*}
  5^2 - 4^2 &= c^2 \\
  25 - 16 &= c^2 \\
  \sqrt{9} &= \sqrt{c^2} \\
  3 &= c \\
  \text{and} \\
  (0, -3) \\
  \text{and} \\
  (0, 3)
\end{align*}
\]

B. \[
\begin{align*}
  4^2 - 1^2 &= c^2 \\
  16 - 1 &= c^2 \\
  \sqrt{15} &= \sqrt{c^2} \\
  3.87 &\approx c \\
  \text{and} \\
  (-1 - \sqrt{15}, 2) \\
  \text{and} \\
  (-1 + \sqrt{15}, 2)
\end{align*}
\]
Eccentricity of an Ellipse is a measure of how close the ellipse is to being a circle and is given by the formula:

\[ \text{Eccentricity} = \frac{c}{a}, \]  

where ‘c’ is the distance from the center to a focal point and ‘a’ is the length of the major radius.

### Sample Ellipses

<table>
<thead>
<tr>
<th>Eccentricity = 0</th>
<th>Eccentricity ≈ 0.75</th>
<th>Eccentricity ≈ 0.94</th>
<th>Eccentricity ≈ 0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Ellipse 1" /></td>
<td><img src="image2.png" alt="Ellipse 2" /></td>
<td><img src="image3.png" alt="Ellipse 3" /></td>
<td><img src="image4.png" alt="Ellipse 4" /></td>
</tr>
</tbody>
</table>

A. Eccentricity = 0

B. Eccentricity ≈ 0.75

C. Eccentricity ≈ 0.94

D. Eccentricity ≈ 0.99

### 7. Find the eccentricity of each of the following ellipses also used in problem number 6.

A. \( a = 5, \quad b = 4, \quad c = 3 \)

\[ \text{Eccentricity} = \frac{c}{a} = \frac{3}{5} \]

\[ = 0.6 \]

\[ E = 0.6 \]

B. \( a = 4, \quad b = 1, \quad c = \sqrt{15} \)

\[ \text{Eccentricity} = \frac{c}{a} = \frac{\sqrt{15}}{4} \]

\[ \approx 0.968 \]

### 8. Find the equations of ellipse given the following parameters and sketch a graph.

**A.** Find the equation and graph of an ellipse that has vertices at \((-2, 5)\) and \((-2, -3)\) and co-vertices at \((-4,1)\) and \((0,1)\).

Center: \(\left(\frac{-2 + 2}{2}, \frac{5 + (-3)}{2}\right) = (-2, 1)\)

\[ \frac{(x+2)^2}{4} + \frac{(y-1)^2}{16} = 1 \]

**B.** Find the equation and graph of an ellipse that has vertices at \((-6, 1)\) and \((4, 1)\) and focal points at \((-5,1)\) and \((3,1)\).

Center: \(\left(\frac{-6 + 4}{2}, \frac{1 + 1}{2}\right) = (-1, 1)\)

\[ \frac{(x+1)^2}{25} + \frac{(y-1)^2}{9} = 1 \]

\[ \frac{a^2}{25} - \frac{b^2}{9} = 1 \]

\[ \frac{a^2}{25} - \frac{b^2}{16} = 1 \]

\[ \frac{a^2}{5^2} - \frac{b^2}{9} = 1 \]

\[ b = 3 \]
9. Consider the following describes the orbit of Halley’s Comet. It orbits the sun in a highly elliptical orbit with the sun being one of the focal points of the ellipse. Determine the following given that \( a = 17.834 \) astronomical units and \( b = 4.534 \) astronomical units.

A. Determine an equation that describes the orbit of Halley’s comet.

\[
\frac{x^2}{17.834^2} + \frac{y^2}{4.534^2} = 1
\]

B. Find the position of the Sun on the coordinate grid.

\[
\begin{align*}
a^2 - b^2 &= c^2 \\
17.834^2 - 4.534^2 &= c^2 \\
\sqrt{17.834^2 - 4.534^2} &= c \\
17.248 &= c
\end{align*}
\]

\((0, 17.248)\)

C. What is the Eccentricity of the orbit?

\[
\frac{c}{a} = \frac{17.248}{17.834} \approx 0.967
\]

10. In an Introduction to Theater Course, a student noticed that the spotlight actually cast a spotlight on the stage floor in the shape of an ellipse. The student laid out a coordinate grid on the floor. The units are in feet.

A. What is the equation of the ellipse created by the spotlight on the coordinate grid?

\[
\frac{x^2}{25} + \frac{y^2}{9} = 1
\]

B. What are the coordinates of the focal points?

\[
\begin{align*}
a^2 - b^2 &= c^2 \\
25 - 9 &= c^2 \\
\sqrt{16} &= \sqrt{c^2} \\
4 &= c
\end{align*}
\]

\((-4, 0)\) and \((4, 0)\)

C. What is the eccentricity of the ellipse?

\[
\frac{c}{a} = \frac{4}{5} = 0.8
\]